Ex1

1) $P_A(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ -4 & 4-\lambda & 0 \\ -2 & 1 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)(-4\lambda + \lambda^2 + 4)$

$= (2-\lambda) * (\lambda - 2)^2 = (2-\lambda)^3$

$\lambda = 2$ v.p. de multiplicité 3.

2) $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ v.p associé à $\lambda = 2$.

$(A - \lambda I)X = 0$ (à d) $\begin{cases} -2x + y = 0 \\ -4x + 2y = 0. \\ -2x + y = 0. \end{cases}$

$y = 2x$; $X = \begin{pmatrix} x \\ 2x \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$; $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ sont 2 v.p associés à $\lambda = 2$

2) $\dim E_\lambda = 2 \neq 3 = \text{mult}(\lambda)$ donc A n'est pas diagonalisable

3) $B = \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix}$; $B^2 = \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$B^n = B^2 \cdot B^{n-2} = 0 \quad \forall n \geq 2.$

4. $A^n = (2I + B)^n = \sum_{k=0}^{n} C_n^k B^k (2I)^{n-k}$

$= (2I)^n + n(2I)^{n-1}B + \underbrace{\cdots +}_{=0}$

$= 2^n I + 2^{n-1} n (A - 2I)$

$= (2^n - 2^n n) I + 2^{n-1} n A$

$= 2^n (1-n) I + 2^{n-1} . n A.$

5. $e^{tA} = I + tA + \frac{(tA)^2}{2!} \cdots$

$= \sum_{n \geq 0} \frac{t^n A^n}{n!} = \sum_{n \geq 0} \frac{t^n}{n!} \left[2^n (1-n) I\right) + 2^{n-1} . n A\right]$

$= \sum_{n \geq 0} \frac{(2t)^n}{n!} I - \sum_{n \geq 1} \frac{(2t)^n}{(n-1)!} I + \sum_{n \geq 1} \frac{2^{n-1} t^n}{(n-1)!} A$

$= e^{2t} I - 2t \sum_{n \geq 1} \frac{(2t)^{n-1}}{(n-1)!} I + t \sum_{n \geq 1} \frac{(2t)^{n-1}}{(n-1)!} A$

$= e^{2t} I - 2t \sum_{n \geq 0} \frac{(2t)^n}{n!} I + t \sum_{n \geq 0} \frac{(2t)^n}{n!} A$

$= (e^{2t} - 2t e^{2t}) I + t e^{2t} . A.$

$= (1 - 2t) e^{2t} I + t . e^{2t} . A.$

() $X = e^{tA} . C$

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$$X = \left[(1-2t)\,e^{2t}\, I + t\,e^{2t} A\right] C \quad , \quad C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} (1-2t)\,e^{2t} + t.e^{2t}\times 0 & t\,e^{2t} & 0 \\ -4t\,e^{2t} & (1-2t)\,e^{2t} + 4t\,e^{2t} & 0 \\ -2t\,e^{2t} & t\,e^{2t} & (1-2t)\,e^{2t} + 2t\,e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\begin{cases} x(t) = c_1(1-2t)\,e^{2t} + c_2\, t\,e^{2t} \\ y(t) = -4c_1\, t\,e^{2t} + c_2(1+2t)\,e^{2t} \\ z(t) = -2c_1\, t\,e^{2t} + c_2\, t\,e^{2t} + c_3\, e^{2t} \end{cases}$$

(20mn)

Ex2 : $f(x,y,z) = x^2 + y^2 + z^2 + xyz$

1) $$\begin{cases} \dfrac{\partial f}{\partial x} = 2x + yz = 0 & (1) \\ \dfrac{\partial f}{\partial y} = 2y + xz = 0 & (2) \\ \dfrac{\partial f}{\partial z} = 2z + xy = 0 & (3) \end{cases}$$

M_1, M_2 vérifient les 3 éq^s donc M_1, M_2 sont 2 pts critiques.

2) (1) $\Rightarrow x = -\dfrac{yz}{2}$

(2) $\Rightarrow 2y - \dfrac{yz^2}{2} = 0$

$y\left(2 - \dfrac{z^2}{2}\right) = 0$

$y = 0$ ou $z^2 = 4 \Rightarrow z = \pm 2$

$y = 0 \Rightarrow x = 0 \Rightarrow z = 0$. donc M_1 pt critique

* $z = -2 \overset{(1)}{\Rightarrow} x = y$

(3) $\Rightarrow -4 + x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$M_3(2, 2, -2)$; $M_4(-2, -2, -2)$

* $z = 2 \Rightarrow x = -y$

(3) $\Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2$

$M_5(2, -2, 2)$; $M_2(-2, 2, 2)$

3) $f(M_i) = 12 - 8 = 4$ $\forall i = 2-5$ donc les M_i ont la m nature

4) $$Hf = \begin{pmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{pmatrix}$$

$$Hf(M_1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$; $\lambda = 2$ est une v. p. de multiplicité 3.

M_1 est un min local.

$$Hf(M_2) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

$$|Hf(M_2) - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & -2 \\ 2 & -2 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & -2 \\ 0 & -4+\lambda & 4-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2-\lambda & 2 & 4 \\ 2 & 2-\lambda & -\lambda \\ 0 & -4+\lambda & 0 \end{vmatrix} = (4-\lambda)\left[\lambda^2 - 2\lambda - 8\right]$$

$$\Delta' = 1 + 8 = 9$$

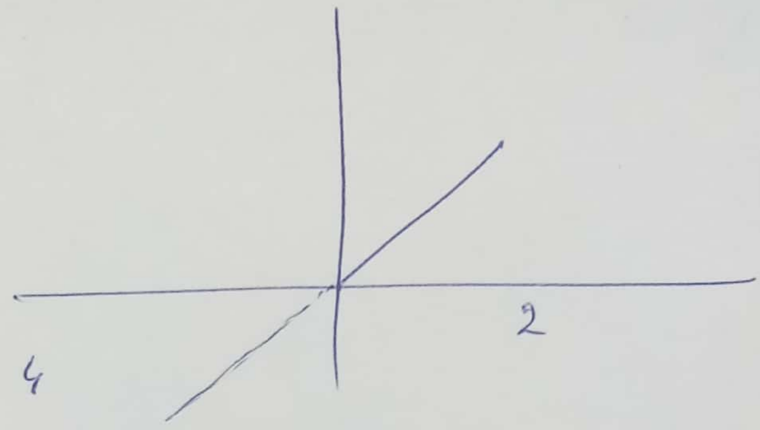
$$\begin{cases} \lambda_1 = \dfrac{-1-3}{1} = -4 \\ \lambda_2 = -1+3 = 2 \\ \lambda_3 = 4 \end{cases}$$

n'ont pas le m̃ signe donc M_2 point selle

5) D'où les autres pts sont selles.

Ex 3

1)



on prolonge $\varphi(x)$ par une fct de période 4 qui soit impaire

$$\tilde{\varphi}(x) = \varphi(x) \text{ sur } [0,2].$$

$$\tilde{\varphi}(x) = \sum_{n\geq 1} b_n \sin(n\omega x) \ ; \ \omega = \frac{\pi}{T} = \frac{\pi}{2}$$

$x \neq 2k.$

$$\varphi(x) = \sum_{n\geq 1} b_n \sin \frac{n\pi}{2} x$$

$$b_n = \frac{2}{2}\int_0^2 x \sin n \frac{\pi}{2} x \, dx$$

$$\begin{array}{ll} x & \sin n \frac{\pi}{2} x \\ & \quad + \\ 1 & -\frac{2}{n\pi} \cos n \frac{\pi}{2} x \\ & \quad - \\ 0 & -\frac{4}{(n\pi)^2} \sin\left(n\frac{\pi}{2} x\right) \end{array}$$

$$b_n = -\frac{2}{n\pi} x \cos n \frac{\pi}{2} x \Big]_0^2 + \frac{4}{(n\pi)^2} \sin n \frac{\pi}{2} x \Big]_0^2$$

$$= -\frac{2}{n\pi} 2 \cos n\pi + \frac{4}{(n\pi)^2} \times 0 = -\frac{4}{n\pi}(-1)^n$$

$$= \frac{4}{n\pi}(-1)^{n-1}$$

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$$\varphi(x) = \frac{4}{\pi}\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sin\left(n\frac{\pi}{2}x\right)$$

2)

a) $\frac{\partial^2 u}{\partial x^2} = f''(x)\, g(t)$; $\frac{\partial u}{\partial t} = f(x) \cdot g'(t)$.

$$f''(x)\, g(t) = 4\, f(x) \cdot g'(t)$$

$$\frac{f''(x)}{f(x)} = \frac{4\, g'(t)}{g(t)} = k$$

$$\begin{cases} f''(x) - k\, f(x) = 0. & (4) \\ g'(t) - \frac{k}{4}\, g(t) = 0 & (5) \end{cases}$$

b) si $k = 0 \Rightarrow f(x) = ax + b$.

$u(x,t) = (ax+b)\, g(t)$.

- $u(0,t) = 0 \Rightarrow b\, g(t) = 0 \Rightarrow b = 0$ car $g(t) \neq 0$.

$u(x,t) = ax\, g(t)$

- $u(2,t) = 0 \Rightarrow 2a\, g(t) = 0 \Rightarrow a = 0$ impossible

donc $k \neq 0$.

si $k = \omega^2 > 0$: $f''(x) - \omega^2 f(x) = 0$.

$\lambda^2 - \omega^2 = 0 \Rightarrow \lambda = \pm\omega$

$f(x) = c_1 e^{-\omega x} + c_2 e^{\omega x}$.

$f(x) = (C_1 e^{-wx} + C_2 e^{wx})\, g(t)$

- $u(0,t) = 0 \Rightarrow (C_1 + C_2)\, g(t) = 0 \Rightarrow C_2 = -C_1$

$u\,\cancel{f}(x,t) = C_1 (e^{-wx} - e^{wx})\, g(t).$

- $u(2,t) = 0 \Rightarrow C_1 (e^{-2w} - e^{2w})\, g(t) = 0$

$w \neq 0 \Rightarrow C_1 = 0 \Rightarrow \cancel{f}\ u(x,t) = 0$;
solut° non interessante

donc $k < 0$.

Soit $k = -w^2$.

$$f''(x) + w^2 f(x) = 0.$$

$$\alpha = iw$$

$$f(x) = C_1 \cos wx + C_2 \sin wx.$$

et $u(x,t) = (C_1 \cos wx + C_2 \sin wx)\, g(t).$

- $u(0,t) = C_1\, g(t) = 0 \Rightarrow C_1 = 0.$

$u(x,t) = C_2 \sin wx\, g(t)$

- $u(2,t) = 0 \Rightarrow C_2 \sin 2w\, g(t) = 0$

$\Rightarrow \sin 2w = 0 = \sin n\pi$

$2w = n\pi \Rightarrow w_n = \frac{n\pi}{2}$

$$\boxed{u_n(x,t) = C_n \sin \frac{n\pi}{2} x\, g(t).}$$

c) $g'(t) + \frac{w^2}{4} g(t) = 0$

$\ln g(t) = -\frac{w^2}{4} t + ct_e$

$g(t) = k e^{-\frac{w^2}{4} t}$

$u_n(x,t) = C_n \sin n\frac{\pi}{2} x \cdot e^{-\frac{n^2\pi^2}{16} t}$

d) ~~u(x,0) = x~~

Soit $u(x,t) = \sum_{n\geq 1} C_n \sin n\frac{\pi}{2} x \, e^{-w_n^2/4\, t}$

$u(x,0) = x \Rightarrow \sum_{n\geq 1} C_n \sin n\frac{\pi}{2} x = x$

D'aprs 1) $C_n = \frac{4}{n\pi} (-1)^{n-1}$

$u(x,t) = \sum_{n\geq 1} \frac{4}{n\pi} (-1)^{n-1} \sin n\frac{\pi}{2} x \, e^{-\frac{n^2\pi^2}{16} t}$.